Matrix product solution to an inhomogeneous multi-species TASEP

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Abstract

We study a multi-species exclusion process with inhomogeneous hopping rates. This model is equivalent to a Markov chain on the symmetric group that corresponds to a random walk in the affine braid arrangement [11]. We find a matrix product representation for the stationary state of this model. We also show that it is equivalent to a graphical construction proposed by Ayyer and Linusson [4], which generalizes Ferrari and Martin's construction [8].

1 Introduction

The coupling of randomness with algebraic or arithmetic structures can lead to beautiful combinatorial results. The study of random partitions of integers and its extension to three-dimensional plane partitions [17, 1, 15] provides a prominent example. Recently, Lam studied Markov chains that can be represented geometrically as random walks on a regular tessellation of a vector space on which an affine Weyl group acts [11]. When this Weyl group is a symmetric group, the stationary distribution of the chain displays remarkable combinatorial properties that were further explored by Lam and Williams, leading to various conjectures [12]. The Markov chain studied in [12] is equivalent to a multi-species exclusion process with inhomogeneous transition rates as shown by Ayyer and Linusson in a very recent publication [4]. This suggests that the powerful techniques that were developed in non-equilibrium statistical mechanics to analyze the asymmetric exclusion process [14, 6, 5] should be relevant to this more mathematical problem. Indeed, Ayyer and Linusson conjectured that the stationary state can be obtained by generalizing the elegant graphical algorithm, invented by Ferrari and Martin [8] to solve the homogeneous N-species totally asymmetric simple exclusion process (N-TASEP); in [4], they generalized Ferrari and Martin's algorithm and explored the consequences for Lam and Williams' conjectures.

In the present work, we solve the N-TASEP with inhomogeneous transition rates by using a generalized matrix product Ansatz. More precisely, we show that a suitable deformation of the homogeneous algebra studied in [16, 2] allows us to calculate the stationary state of the inhomogeneous N-TASEP. We also show that our matrix product solution is equivalent to the algorithm conjectured in [4].

The outline of this work is as follows. In section 2, the model is defined. In section 3, we give a matrix product solution to the stationary state. In section 4, we explain the graphical construction of the stationary state conjectured by Ayyer and Linusson, and we show that this construction is equivalent to the matrix product solution. We give concluding remarks in section 5.

2 Definition of the model

We consider an L-site periodic chain in which each site takes a non-negative integer value $1, 2, \dots,$ or $N+1$ (this is called the N-*species* problem). Each pair of nearest neighbor sites exchanges their values according to the following continuous time stochastic dynamics:

$$a\ b\ \to b\ a \quad \text{with rate} \begin{cases} x_a & (a<b) \\ 0 & (\text{otherwise}) \end{cases} \tag{2.1}$$

The special case $x_a = 1$ for all a corresponds to the *homogeneous* N-TASEP [2, 3, 8, 7]. When the transition rates x_a's are not equal to each other, we say that this dynamics is *inhomogeneous* (although one could consider even more complicated or general transition rules). When $L = N + 1$, and there is exactly one particle of each species in the system, the process is equivalent to Lam and Williams' process on the symmetric group S_L [12].

The inhomogeneous N-TASEP is governed by the master equation

$$\frac{d}{dt}|P\rangle = M^{(N)}|P\rangle \tag{2.2}$$

for the vector $|P\rangle = \sum P(j_1 \cdots j_L)|j_1 \cdots j_L\rangle$, where $P(j_1 \cdots j_L)$ is the probability of finding the system in a configuration $j_1 \cdots j_L$. The generator matrix (Markov matrix) $M^{(N)}$ is the summation of local operators $\left(M^{(N)}_{\mathrm{Loc}}\right)_{i,i+1}$ that acts on the spaces corresponding to ith and $(i+1)$st sites of the chain:

$$M^{(N)} = \sum_{i=1}^{L} \left(M^{(N)}_{\mathrm{Loc}}\right)_{i,i+1}, \; M^{(N)}_{\mathrm{Loc}} = \sum_{a,b=1}^{N+1} \Theta(a-b)\Big(|ba\rangle\langle ab| - |ab\rangle\langle ab|\Big) \text{ with } \Theta(a-b) = \begin{cases} x_a & (a < b), \\ 0 & (a \geq b). \end{cases} \tag{2.3}$$

We write $m = (m_1, \dots, m_{N+1})$ (with $m_i \in \mathbb{Z}_{\geq 0}$) for the sector that contains m_k particles of type k. Because of the conservation of the number of particles of each type, we have the decomposition $M^{(N)} = \bigoplus_m M_m$. In particular, we consider *basic sectors*, i.e. $m_i > 0$. Since the identification of local states $N + 1 \to N$ maps the N-species dynamics to the $(N-1)$-species dynamics, we have the spectral inclusion: the spectrum of the sector $m = (m_1, \dots, m_N, m_{N+1})$ contains that of the sector $m' = (m_1, \dots, m_N + m_{N+1})$.[1] This inclusion relation indicates that there exists a "conjugation matrix" ψ_m such that

$$M_m \psi_m = \psi_m M_{m'}. \tag{2.4}$$

This matrix allows one to lift states that belong to the $(N-1)$-species sector m' and to construct eigenstates for the N-species sector m. In particular, the stationary state of the N-TASEP (the kernel of $M^{(N)}$) can be constructed recursively if one knows the sequence of the conjugation matrices.

3 Generalized matrix product Ansatz

The idea of the matrix product Ansatz is to express the probability of finding each configuration $j_1 \cdots j_L$ as a trace over a suitable algebra

$$P(j_1 \cdots j_L) = \frac{1}{Z}\mathrm{Tr}\Big(X^{(N)}_{j_1} \cdots X^{(N)}_{j_L}\Big) \tag{3.1}$$

with a normalization constant Z. The operators $X^{(N)}_j$'s must satisfy suitable algebraic relations, which are usually infinite dimensional [6, 5]. For the homogeneous N-TASEP, the operators $X^{(N)}_j$ were constructed recursively [7, 16] as tensor products of four fundamental operators δ, ϵ, A and $\mathbb{1}$ that act on an infinite dimensional space $\mathcal{A} = \bigotimes_{\mu \geq 0} \mathbb{C}|\mu\rangle\rangle$ as

$$\delta|\mu\rangle\rangle = \begin{cases} 0 & (\mu = 0), \\ |\mu - 1\rangle\rangle & (\mu > 0), \end{cases} \quad \epsilon|\mu\rangle\rangle = |\mu + 1\rangle\rangle, \; A|\mu\rangle\rangle = \begin{cases} |0\rangle\rangle & (\mu = 0), \\ 0 & (\mu > 0), \end{cases} \quad \mathbb{1}|\mu\rangle\rangle = |\mu\rangle\rangle. \tag{3.2}$$

One can easily verify that the following quadratic relations are satisfied:

$$\delta\epsilon = \mathbb{1}, \; \delta A = 0, \; A\epsilon = 0\,. \tag{3.3}$$

In [16], the stationary state for the homogeneous N-TASEP has been expressed as a matrix product form, using an algebraic interpretation of Ferrari and Martin's algorithm [8]. More recently the matrix product Ansatz

[1] More general inclusion relations are satisfied for the homogeneous case, see [3].

technique was generalized to obtain a conjugation matrix that satisfies "conjugation relation" (2.4) between systems having different numbers of species [2]. We now explain how the same ideas can be adapted to the inhomogeneous case.

The stationary state for $N = 1$ case is trivial, i.e. all the possible states in each sector are realized with a same probability. This can be regarded as one dimensional representation of the matrices $X_1^{(1)} = X_2^{(1)} = 1$.

3.1 The 2 species case

For $N = 2$, a matrix product stationary representation has been known, even in the inhomogeneous case [5]. One possible choice for the algebra is

$$X_1^{(2)} X_3^{(2)} = \frac{1}{x_1} X_1^{(2)} + X_3^{(2)}, \quad X_2^{(2)} X_3^{(2)} = \frac{1}{x_2} X_2^{(2)}, \quad X_1^{(2)} X_2^{(2)} = X_2^{(2)}. \tag{3.4}$$

Defining an operator-valued vector $\mathbf{X}^{(2)} = \begin{pmatrix} X_1^{(2)} \\ X_2^{(2)} \\ X_3^{(2)} \end{pmatrix}$, we observe that the following decomposition exists

$$\mathbf{X}^{(2)} = \begin{pmatrix} \mathbb{1} & \delta \\ 0 & A \\ y_1 \epsilon & y_1 B + y_2 A \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{3.5}$$

where we have set $y_1 = 1/x_1, y_2 = 1/x_2$ and the matrices δ, ϵ, A satisfy (3.3). This decomposition can be written more formally as the product of two rectangular operator-valued matrices $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$ of sizes 2×1 and 3×2 respectively

$$\mathbf{X}^{(2)} = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} \\ a_{31}^{(2)} & a_{32}^{(2)} \end{pmatrix} \begin{pmatrix} a_{11}^{(1)} \\ a_{21}^{(1)} \end{pmatrix} := \mathbf{a}^{(2)} \star \mathbf{a}^{(1)}, \tag{3.6}$$

where the symbol $\star$ means that we perform tensor products amongst the elements of the matrices $\mathbf{a}^{(2)}$ and $\mathbf{a}^{(1)}$. More generally, for matrix-valued matrices $Y = (Y_{uv})_{uv}, Z = (Z_{uv})_{uv}$, it represents the product $Y \star Z = \left(\sum_w Y_{uw} \otimes Z_{wv}\right)_{uv}$. Since $a_{11}^{(1)}$ and $a_{21}^{(1)}$ are scalars here, the tensor product reduces to the ordinary product. Finally, we observe that the matrices $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$ satisfies

$$M_{\text{Loc}}^{(1)} \mathbf{a}^{(1)} \otimes \mathbf{a}^{(1)} - \mathbf{a}^{(1)} \otimes \mathbf{a}^{(1)} M_{\text{Loc}}^{(0)} = \widehat{\mathbf{a}}^{(1)} \otimes \mathbf{a}^{(1)} - \mathbf{a}^{(1)} \otimes \widehat{\mathbf{a}}^{(1)} \tag{3.7}$$

$$M_{\text{Loc}}^{(2)} \mathbf{a}^{(2)} \otimes \mathbf{a}^{(2)} - \mathbf{a}^{(2)} \otimes \mathbf{a}^{(2)} M_{\text{Loc}}^{(1)} = \widehat{\mathbf{a}}^{(2)} \otimes \mathbf{a}^{(2)} - \mathbf{a}^{(2)} \otimes \widehat{\mathbf{a}}^{(2)} \tag{3.8}$$

where

$$\widehat{\mathbf{a}}^{(1)} = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \quad \widehat{\mathbf{a}}^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \epsilon & \mathbb{1} \end{pmatrix}. \tag{3.9}$$

Since $M_{\text{Loc}}^{(0)} = 0$, equation (3.7) is the relation in the usual matrix product Ansatz [5]. The relation (3.8) implies that the matrix ψ_m whose elements are given as

$$\langle j_1 \cdots j_L | \psi_m | k_1 \cdots k_L \rangle = \mathrm{Tr}\left(a_{j_1 k_1}^{(2)} \cdots a_{j_L k_L}^{(2)} \right) \tag{3.10}$$

intertwines the dynamics of the sectors $m = (m_1, m_2, m_3)$ and $m' = (m_1, m_2 + m_3)$, i.e. it satisfies equation (2.4). Formally we write the stationary state for the sector m' as $\psi_{m'}$ with $\langle j_1 \cdots j_L | \psi_{m'} | 1 \cdots 1 \rangle = \mathrm{Tr}\left(a_{j_1 1}^{(1)} \cdots a_{j_L 1}^{(1)} \right) = 1$. Then the stationary state for the sector m can be rewritten as $\psi_m \psi_{m'}$.

For a simple nontrivial example, the stationary state of the sector $(1,1,2)$ is given as

$$\psi_{(1,1,2)}\psi_{(1,3)} = \begin{array}{c c} & \begin{array}{cccc} 1222 & 2122 & 2212 & 2221 \end{array} \\ \begin{array}{c} 1233 \\ 1323 \\ 1332 \\ 2133 \\ 2313 \\ 2331 \\ 3123 \\ 3132 \\ 3213 \\ 3231 \\ 3312 \\ 3321 \end{array} & \begin{pmatrix} y_2^2 & 0 & 0 & 0 \\ y_2^2 & y_1y_2 & 0 & 0 \\ y_2^2 & y_1y_2 & y_1^2 & 0 \\ 0 & y_2^2 & y_1y_2 & y_1^2 \\ 0 & 0 & y_2^2 & y_1y_2 \\ 0 & 0 & 0 & y_2^2 \\ 0 & y_2^2 & 0 & 0 \\ 0 & y_2^2 & y_1y_2 & 0 \\ y_1^2 & 0 & y_2^2 & y_1y_2 \\ y_1y_2 & 0 & 0 & y_2^2 \\ 0 & 0 & y_2^2 & 0 \\ y_1y_2 & y_1^2 & 0 & y_2^2 \end{pmatrix} \end{array} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} y_2^2 \\ y_2(y_1+y_2) \\ y_1^2+y_1y_2+y_2^2 \\ y_1^2+y_1y_2+y_2^2 \\ y_2(y_1+y_2) \\ y_2^2 \\ y_2^2 \\ y_2(y_1+y_2) \\ y_1^2+y_1y_2+y_2^2 \\ y_2(y_1+y_2) \\ y_2^2 \\ y_1^2+y_1y_2+y_2^2 \end{pmatrix}. \tag{3.11}$$

3.2 The N-species case

We now explain the generalized matrix product Ansatz for general values of N (see [2] for more details). Reversing the construction in the last subsection, we start with the following relation that we call "hat relation":

$$M_{\mathrm{Loc}}^{(N)}\mathbf{a}^{(N)}\otimes\mathbf{a}^{(N)} - \mathbf{a}^{(N)}\otimes\mathbf{a}^{(N)}M_{\mathrm{Loc}}^{(N-1)} = \widehat{\mathbf{a}}^{(N)}\otimes\mathbf{a}^{(N)} - \mathbf{a}^{(N)}\otimes\widehat{\mathbf{a}}^{(N)}, \tag{3.12}$$

where $\mathbf{a}^{(N)}$ and $\widehat{\mathbf{a}}^{(N)}$ are operator-valued matrices of size $(N+1)\times N$. We write their elements as $\langle j|\mathbf{a}^{(N)}|k\rangle = a_{jk}^{(N)}$, $\langle j|\widehat{\mathbf{a}}^{(N)}|k\rangle = \widehat{a}_{jk}^{(N)}$. We know that, if we can construct a couple $\{\mathbf{a}^{(N)}, \widehat{\mathbf{a}}^{(N)}\}$ (for the general integer of N) that satisfies (3.12), the matrix ψ_m defined as

$$\langle j_1\cdots j_L|\psi_m|k_1\cdots k_L\rangle = \mathrm{Tr}\Big(a_{j_1k_1}^{(N)}\cdots a_{j_Lk_L}^{(N)}\Big) \tag{3.13}$$

satisfies the conjugation relation (2.4). Here the configurations $j_1\cdots j_L$ and $k_1\cdots k_L$ belong to the sectors $m=(m_1,\ldots,m_{N+1})$ and $m'=(m_1,\ldots,m_N+m_{N+1})$, respectively. Furthermore, the stationary state of the sector m can be written by the product of conjugation matrices

$$|\bar{P}\rangle_m = \psi_m\psi_{m'}\cdots\psi_{(m_1,L-m_1)} \tag{3.14}$$

if all of them are nonzero (we note that the conjugation matrix ψ_m lifts up other eigenstates from lower sectors as well).

We set $y_i = 1/x_i$ and define the operator $B = \mathbb{1} - A$. An explicit solution to the hat relation (3.12) is given by

$$a_{jk}^{(N)} = \begin{cases} A^{\otimes(j-1)}\otimes\delta\otimes\mathbb{1}^{\otimes(k-j-1)}\otimes\epsilon\otimes\mathbb{1}^{\otimes(N-k-1)} & (j<k<N), \\ A^{\otimes(j-1)}\otimes\delta\otimes\mathbb{1}^{\otimes(N-j-1)} & (j<k=N), \\ A^{\otimes(j-1)}\otimes\mathbb{1}^{\otimes(N-j)} & (j=k), \\ \Big(\sum_{i=1}^{k-1} y_iA^{\otimes(i-1)}\otimes B\otimes\mathbb{1}^{\otimes(k-i-1)} + y_kA^{\otimes(k-1)}\Big)\otimes\epsilon\otimes\mathbb{1}^{\otimes(N-k-1)} & (k<j-1=N), \\ \sum_{i=1}^{N-1} y_iA^{\otimes(i-1)}\otimes B\otimes\mathbb{1}^{\otimes(N-i-1)} + y_NA^{\otimes(N-1)} & (j-1=k=N), \\ 0 & (\text{otherwise}) \end{cases} \tag{3.15}$$

$$\widehat{a}_{jk}^{(N)} = \begin{cases} \mathbb{1}^{\otimes(k-1)}\otimes\epsilon\otimes\mathbb{1}^{\otimes(N-k-1)} & (k<j-1=N) \\ \mathbb{1}^{\otimes(N-1)} & (k=j-1=N) \\ 0 & (\text{otherwise}). \end{cases} \tag{3.16}$$

The difference from the homogeneous case appears in $a_{jk}^{(N)}$ for $j=N+1$. Indeed we retrieve the solution to the homogeneous case [2, 7] by setting $y_j = 1$. We change the definitions of $\mathbf{a}^{(1)}$ and $\widehat{\mathbf{a}}^{(1)}$ as $\mathbf{a}^{(1)} = \begin{pmatrix} 1 \\ y_1 \end{pmatrix}$, $\widehat{\mathbf{a}}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for compatibility with the general forms (3.15), (3.16) .

By a direct calculation, one can show that equations (3.15), (3.16) give a representation to the algebra defined by the hat relation (3.12) i.e.

$$-x_j a_{jk}a_{j'k'} - x_k(a_{jk'}a_{j'k} - a_{jk}a_{j'k'}) = \widehat{a}_{jk}a_{j'k'} - a_{jk}\widehat{a}_{j'k'} \quad (j < j' \wedge k < k'), \tag{3.17}$$
$$-x_k(a_{jk'}a_{j'k} - a_{jk}a_{j'k'}) = \widehat{a}_{jk}a_{j'k'} - a_{jk}\widehat{a}_{j'k'} \quad (j = j' \wedge k < k'), \tag{3.18}$$
$$x_{j'}a_{j'k}a_{jk'} - x_k(a_{jk'}a_{j'k} - a_{jk}a_{j'k'}) = \widehat{a}_{jk}a_{j'k'} - a_{jk}\widehat{a}_{j'k'} \quad (j > j' \wedge k < k'), \tag{3.19}$$
$$-x_j a_{jk}a_{j'k'} = \widehat{a}_{jk}a_{j'k'} - a_{jk}\widehat{a}_{j'k'} \quad (j < j' \wedge k \geq k'), \tag{3.20}$$
$$0 = \widehat{a}_{jk}a_{j'k'} - a_{jk}\widehat{a}_{j'k'} \quad (j = j' \wedge k \geq k'), \tag{3.21}$$
$$x_{j'}a_{j'k}a_{jk'} = \widehat{a}_{jk}a_{j'k'} - a_{jk}\widehat{a}_{j'k'} \quad (j > j' \wedge k \geq k'). \tag{3.22}$$

It is straightforward to prove that the above relations are satisfied simply by substituting equations (3.15) and (3.16). In appendix, we prove the first identity (3.17) as an example. For $N = 3$ and 4, the result can be written explicitly as

$$\mathbf{a}^{(3)} = \begin{pmatrix} \mathbb{1}\otimes\mathbb{1} & \delta\otimes\epsilon & \delta\otimes\mathbb{1} \\ 0 & A\otimes\mathbb{1} & A\otimes\delta \\ 0 & 0 & A\otimes A \\ y_1\epsilon\otimes\mathbb{1} & \begin{array}{c} y_1B\otimes\epsilon \\ +y_2A\otimes\epsilon \end{array} & \begin{array}{c} y_1B\otimes\mathbb{1} \\ +y_2A\otimes B \\ +y_3A\otimes A \end{array} \end{pmatrix}, \quad \widehat{\mathbf{a}}^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \epsilon\otimes\mathbb{1} & \mathbb{1}\otimes\epsilon & \mathbb{1}\otimes\mathbb{1} \end{pmatrix}, \tag{3.23}$$

$$\mathbf{a}^{(4)} = \begin{pmatrix} \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} & \delta\otimes\epsilon\otimes\mathbb{1} & \delta\otimes\mathbb{1}\otimes\epsilon & \delta\otimes\mathbb{1}\otimes\mathbb{1} \\ 0 & A\otimes\mathbb{1}\otimes\mathbb{1} & A\otimes\delta\otimes\epsilon & A\otimes\delta\otimes\mathbb{1} \\ 0 & 0 & A\otimes A\otimes\mathbb{1} & A\otimes A\otimes\delta \\ 0 & 0 & 0 & A\otimes A\otimes A \\ y_1\epsilon\otimes\mathbb{1}\otimes\mathbb{1} & \begin{array}{c} y_1B\otimes\epsilon\otimes\mathbb{1} \\ +y_2A\otimes\epsilon\otimes\mathbb{1} \end{array} & \begin{array}{c} y_1B\otimes\mathbb{1}\otimes\epsilon \\ +y_2A\otimes B\otimes\epsilon \\ +y_3A\otimes A\otimes\epsilon \end{array} & \begin{array}{c} y_1B\otimes\mathbb{1}\otimes\mathbb{1} \\ +y_2A\otimes B\otimes\mathbb{1} \\ +y_3A\otimes A\otimes B \\ +y_4A\otimes A\otimes A \end{array} \end{pmatrix}, \tag{3.24}$$

$$\widehat{\mathbf{a}}^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon\otimes\mathbb{1}\otimes\mathbb{1} & \mathbb{1}\otimes\epsilon\otimes\mathbb{1} & \mathbb{1}\otimes\mathbb{1}\otimes\epsilon & \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} \end{pmatrix}. \tag{3.25}$$

We now generalize the form (3.5) or (3.6). The form (3.14) can be written as the matrix product form (3.1) with the matrices

$$X_j^{(N)} = \langle j|X^{(N)}|1\rangle, \quad X^{(N)} = \mathbf{a}^{(N)} \star \cdots \star \mathbf{a}^{(1)}. \tag{3.26}$$

thanks to the "sector specificity"[2].

Let us consider the element (3.13) for configurations $j_1 \cdots j_L$ and $k_1 \cdots k_L$ of the sectors $(m_1, \ldots, m_{N+1})$ and $(m_1, \ldots, m_N + m_{N+1})$ with $j_i < j_{i+1}$. Since $a_{jk}a_{j'j'} = 0$ for $j < j' \leq N$ and $k \neq k'$, we need to set $j_i = k_i$ for $i \leq L - m_{N+1}$ so that (3.13) is nonzero, and we have

$$\mathrm{Tr}\left(a_{11} \cdots a_{11}a_{22} \cdots a_{22} \cdots a_{NN} \cdots a_{NN}a_{(N+1)N} \cdots a_{(N+1)N}\right) = y_N^{m_{N+1}}. \tag{3.27}$$

Therefore the stationary weight of the configuration $j_1 \cdots j_L$ is $\prod_{n=1}^N y_n^{m_{n+1}+\cdots+m_{N+1}}$. As far as we treat basic sectors, the matrix product $a_{j_1k_1}^{(N)} \cdots a_{j_Lk_L}^{(N)}$ contains $a_{NN}^{(N)} = A^{\otimes(N-1)}$ or 0. This implies that the matrix product is transformed into a tensor product of the form $\epsilon^c A\delta^c = |c\rangle\!\rangle\langle\!\langle c|$ multiplied by a monomial of y_n's (by applying the relation (3.3) and $A^2 = A$), otherwise it is 0. Since $\mathrm{Tr}\epsilon^c A\delta^c = 1$, the trace of $a_{j_1k_1}^{(N)} \cdots a_{j_Lk_L}^{(N)}$ is a monomial of y_n's or 0. Note that the trace is not always finite if we consider a *non* basic sector.

[2] When $k_1 \cdots k_L$ does not belong to the sector m', the trace (3.13) is always 0. This specificity is because the numbers of δ's and ϵ's are different in the matrix product, see [2] for details.

4 Graphical construction of the stationary state

We have found a solution to the generalized matrix product Ansatz that constructs conjugation matrices of the inhomogeneous N-TASEP. In this section, we first review the algorithm that A. Ayyer and S. Linusson devised [4] to calculate the stationary weights by defining an inhomogeneous extension of the seminal algorithm by Ferrari and Martin[8]. Then we show that the solution to the matrix product Ansatz of the previous section is equivalent to Ayyer and Linusson's construction.

4.1 Ayyer and Linusson's algorithm

The algorithm of Ayyer and Linusson [4] constructs the stationary state of the N-species sector $(m_1, \ldots, m_{N+1})$ from the $(N-1)$-species sector $(m_1, \ldots, m_N + m_{N+1})$. It is provided by two maps F, W from an $(N-1)$-species configuration and a configuration consisting of black and white boxes, to an N-species configuration and a polynomial in y_i's. Figure 4.1 is helpful to understand the algorithm.

(i) Let us set two lines. On the upper line, there are $m_1 + \cdots + m_N$ black boxes $\blacksquare$ and m_{N+1} white boxes $\square$ as $c_1 \cdots c_L$ ($c_i = \blacksquare, \square$). On the lower line, we give a configuration $k_1 \cdots k_L$ of the $(N-1)$-species sector $(m_1, \ldots, m_N + m_{N+1})$. Thus, on the lower line, there are m_ν ν's $(1 \le \nu \le N-1)$ and $(m_N + m_{N+1})$ N's.

(ii-1) Let $\{i_1^{(1)}, \ldots, i_{m_1}^{(1)}\}$ be the positions of the m_1 1's on the lower line. For the first 1 located at $i_1^{(1)}$, find the nearest black box $c_{i'} = \blacksquare$ with $i' \le i_1^{(1)}$ and put 1 on it. If there is no such black box, put 1 on the rightmost black box. For the second 1 located at $i_2^{(1)}$, find the nearest unoccupied black box $c_{i'} = \blacksquare$ with $i' \le i_2^{(1)}$ and put 1 on it. If there is no such black box, put 1 on the rightmost unoccupied black box. We draw an arrow from $i_1^{(1)}$ to the targeted black box. (In the following procedure till (ii-ν), we always draw an arrow in the same way, see figure (4.1)) We iterate this procedure m_1 times, i.e. find the nearest unoccupied black box $c_{i'} = \blacksquare$ with $i' \le i_\ell^{(1)}$ for the ℓth 1 located at $i_\ell^{(1)}$, and put 1 on it or on the rightmost unoccupied black box if i' does not exist.

(ii-2) Let $\{i_1^{(2)}, \ldots, i_{m_2}^{(2)}\}$ be the positions of the m_2 2's on the lower line. There are $(m_2 + \cdots + m_N)$ unoccupied black boxes on the upper line. We iterate the following procedure m_2 times: find the nearest unoccupied black box $c_{i'} = \blacksquare$ with $i' \le i_\ell^{(2)}$ for the ℓth 2 $(1 \le \ell \le m_2)$, and put 2 on it or on the rightmost unoccupied black box if i' does not exist.

(ii-ν) In the same way, we go on for $\nu = 3, 4, \ldots, (N-1)$. Let $\{i_1^{(\nu)}, \ldots, i_{m_\nu}^{(\nu)}\}$ be the positions of the m_ν ν's on the lower line. There are $(m_\nu + \cdots + m_N)$ unoccupied black boxes remaining on the upper line. We iterate the following procedure m_ν times: find the nearest unoccupied black box $c_{i'} = \blacksquare$ with $i' \le i_\ell^{(\nu)}$ for the ℓth ν $(1 \le \ell \le m_\nu)$, and put ν on it or on the rightmost unoccupied black box if i' does not exist.

(iii) There are m_N unoccupied black boxes remaining. Put Ns on them.

(iv) Put $(N+1)$s on the m_{N+1} white boxes. We have thus constructed a configuration $F(c_1 \cdots c_L, k_1 \cdots k_L)$ of the N-TASEP on the upper line, belonging to the sector m.

(v) Define a vector from the stationary state $|\bar{P}_{m'}\rangle$ of the sector m' as follows:

$$|\bar{P}_m\rangle = \sum W(j_1 \cdots j_L, k_1 \cdots k_L) |j_1 \cdots j_L\rangle \langle k_1 \cdots k_L | \bar{P}_{m'}\rangle. \tag{4.1}$$

The summation $\sum$ runs over $j_1 \cdots j_L$ and $k_1 \cdots k_L$ belonging to the sectors m and m'. If there exits a configuration $c_1 \cdots c_L$ such that

$$F(c_1 \cdots c_L, k_1 \cdots k_L) = j_1 \cdots j_L, \tag{4.2}$$

the coefficient $W(j_1 \cdots j_L, k_1 \cdots k_L)$ is defined as the product of the following weights $\{w_1, \cdots, w_L\}$. Draw a vertical line between each bond between site $i-1$ and i, see figure 4.1. When an arrow connecting two

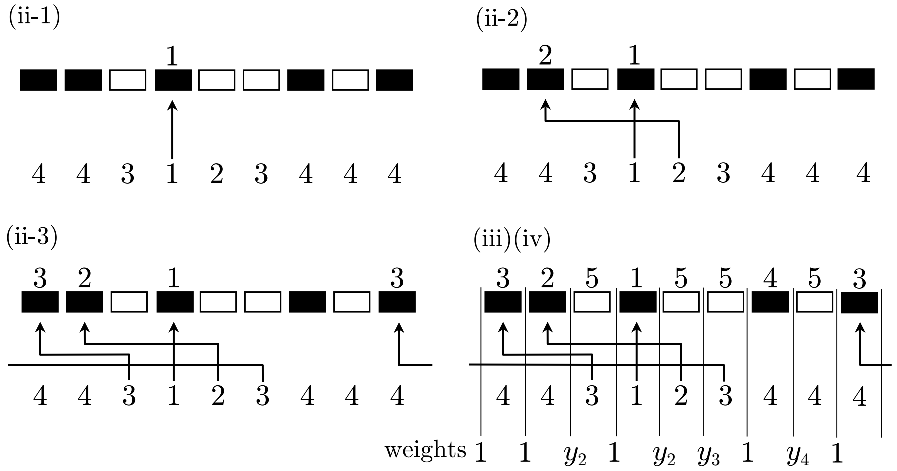

Figure 1: An example of Ayyer and Linusson's algorithm.

ν's on the upper and lower lines in our figures, we say "the value of the arrow is ν": $\overset{\nu}{\underset{\nu}{\hookleftarrow}}$. Each weight w_i is defined as

$$w_i = \begin{cases} 1 & (j_i \le N), \\ y_\ell & (j_i = N+1, \text{ and } \ell \text{ is the minimal value of arrows that cross the } i\text{th vertical line}), \\ y_N & (j_i = N+1, \text{ and no arrow crosses the } i\text{th vertical line}). \end{cases} \tag{4.3}$$

Then we have $W(j_1 \cdots j_L, k_1 \cdots k_L) = \prod_{1 \le i \le L} w_i$. If there is no configuration $c_1 \cdots c_L$ such that equation (4.2) is satisfied, we define $W(j_1 \cdots j_L, k_1 \cdots k_L) = 0$.

Ayyer and Linusson conjectured that the form (4.1) gives the stationary state $|\bar{P}_m\rangle$ of the sector m [4]. There the definition of the weight looks different from our W, but is equivalent by multiplying it by a constant.

Figure 4.1 gives an example of the algorithm for the sectors $m = (1,1,2,1,4), m' = (1,1,2,5)$, where the upper and lower lines are

$$c_1 \cdots c_L = ■■□■□□■□■, \quad k_1 \cdots k_L = 443123444. \tag{4.4}$$

According to the algorithm, the configuration of the sector m and the weight are obtained as

$$F(■■□■□□■□■, 443123444) = 325155453, \quad W(325155453, 443123444) = y_2^2 y_3 y_4. \tag{4.5}$$

4.2 Equivalence of the matrix representation and Ayyer and Linusson's algorithm

We explain the relation between the matrix product representation and Ayyer and Linusson's algorithm (following [2] for the homogeneous case). The elements $a_{jk}^{(N)}$ act on a basis vector $|\mu_1, \ldots, \mu_{N-1}\rangle\rangle = |\mu_1\rangle\rangle \otimes \cdots \otimes |\mu_{N-1}\rangle\rangle \in$

$\mathcal{A}^{\otimes(N-1)}$ as

$$a_{jk}^{(N)}|\mu_1,\dots,\mu_{N-1}\rangle\!\rangle$$
$$=\begin{cases} |\mu_1,\dots,\mu_j-1,\dots,\mu_k+1,\dots,\mu_{N-1}\rangle\!\rangle & (j<k<N, \mu_1=\cdots=\mu_{j-1}=0, \mu_j>0),\\ |\mu_1,\dots,\mu_j-1,\dots,\mu_{N-1}\rangle\!\rangle & (j<k=N, \mu_1=\cdots=\mu_{j-1}=0, \mu_j>0),\\ |\mu_1,\dots,\mu_{N-1}\rangle\!\rangle & (j=k, \mu_1=\cdots=\mu_{j-1}=0),\\ y_{\min\{\ell,k\}}|\mu_1,\dots,\mu_k+1,\dots,\mu_{N-1}\rangle\!\rangle & (k<j-1=N, \mu_1=\cdots=\mu_{\ell-1}=0, \mu_\ell>0),\\ y_\ell|\mu_1,\dots,\mu_{N-1}\rangle\!\rangle & (j-1=k=N, \mu_1=\cdots=\mu_{\ell-1}=0, \mu_\ell>0),\\ 0 & (\text{otherwise}). \end{cases} \tag{4.6}$$

Let $j_1\cdots j_L$ and $k_1\cdots k_L$ belong to basic sectors m and m'. If the vector $|\mu_1,\dots,\mu_{N-1}\rangle\!\rangle$ is not killed by a matrix product $a_{j_1k_1}\cdots a_{j_Lk_L}$, the series of matrices $\{a_{j_1k_1},\dots,a_{j_Lk_L}\}$ give a series ("trajectory") as

$$\begin{aligned} &|\mu_1,\dots,\mu_{N-1}\rangle\!\rangle \overset{a_{j_Lk_L}^{(N)}}{\longmapsto} v_L|\mu_1^L,\dots,\mu_{N-1}^L\rangle\!\rangle \overset{a_{j_{L-1}k_{L-1}}^{(N)}}{\longmapsto} v_{L-1}v_L|\mu_1^{L-1},\dots,\mu_{N-1}^{L-1}\rangle\!\rangle \mapsto \\ &\cdots \mapsto v_2\cdots v_L|\mu_1^2,\dots,\mu_{N-1}^2\rangle\!\rangle \overset{a_{j_1k_1}^{(N)}}{\longmapsto} v_1\cdots v_L|\mu_1^1,\dots,\mu_{N-1}^1\rangle\!\rangle, \end{aligned} \tag{4.7}$$

where we have set

$$a_{j_ik_i}^{(N)}|\mu_1^{i+1},\dots,\mu_{N-1}^{i+1}\rangle\!\rangle = v_i|\mu_1^i,\dots,\mu_{N-1}^i\rangle\!\rangle \quad (\mu_\nu^{L+1}=\mu_\nu). \tag{4.8}$$

Since $a_{jk}^{(N)}$ increases $\mu_k\mapsto\mu_k+1$ and decreases $\mu_j\mapsto\mu_j-1$ (see the action (4.6)), and we have $\#\{j_i=\nu\}=\#\{k_i=\nu\}$ for $\nu\le N-1$, we find $|\mu_1^1,\dots,\mu_{N-1}^1\rangle\!\rangle=|\mu_1,\dots,\mu_{N-1}\rangle\!\rangle$. In other words, $|\mu_1,\dots,\mu_{N-1}\rangle\!\rangle$ is an eigenvector of the matrix product with a nonzero eigenvalue. If a trajectory (4.7) is given, one notices that it is unique and $a_{j_1k_1}^{(N)}\cdots a_{j_Lk_L}^{(N)}|\mu_1',\dots,\mu_{N-1}'\rangle\!\rangle=0$ for $(\mu_1',\dots,\mu_{N-1}')\neq(\mu_1,\dots,\mu_{N-1})$. We regard μ_ν in the vector $|\mu_1,\dots,\mu_{N-1}\rangle\!\rangle$ as the number of arrows with value ν. The trajectory gives one graph of arrows which is the same as obtained by Ayyer and Linusson's algorithm since the local action (4.6) is compatible with it. (For example, when $j<k<N$, the action of $a_{jk}^{(N)}$ decreases # of arrows with value j and increases # of arrows with value k. This is allowed only if there is no arrow with value $\nu<j$, otherwise it kills the vector.) We can also see a compatibility of the local action for the coefficient $v_i=w_i$ as well as arrows, and thus the nonzero eigenvalue is identical to $W(j_1\cdots j_L,k_1\cdots k_L)$:

$$\mathrm{Tr}\left(a_{j_1k_1}^{(N)}\cdots a_{j_Lk_L}^{(N)}\right)=W(j_1\cdots j_L,k_1\cdots k_L). \tag{4.9}$$

When there is no nonzero eigenvalue, this equation is also true since $W(j_1\cdots j_L,k_1\cdots k_L)=0$.

For example, for configurations $j_1\cdots j_L=325155453$, $k_1\cdots k_L=443123444$ as in figure 4.1, the matrix product $a_{j_1k_1}\cdots a_{j_Lk_L}$ gives a trajectory

$$\begin{aligned} &|0,0,1\rangle\!\rangle \overset{a_{34}^{(4)}}{\longmapsto} |0,0,0\rangle\!\rangle \overset{a_{54}^{(4)}}{\longmapsto} y_4|0,0,0\rangle\!\rangle \overset{a_{44}^{(4)}}{\longmapsto} |0,0,0\rangle\!\rangle \overset{a_{53}^{(4)}}{\longmapsto} y_3y_4|0,0,1\rangle\!\rangle \overset{a_{52}^{(4)}}{\longmapsto} y_2y_3y_4|0,1,1\rangle\!\rangle \\ &\overset{a_{11}^{(4)}}{\longmapsto} y_2y_3y_4|0,1,1\rangle\!\rangle \overset{a_{53}^{(4)}}{\longmapsto} y_2^2y_3y_4|0,1,2\rangle\!\rangle \overset{a_{24}^{(4)}}{\longmapsto} y_2^2y_3y_4|0,0,2\rangle\!\rangle \overset{a_{34}^{(4)}}{\longmapsto} y_2^2y_3y_4|0,0,1\rangle\!\rangle, \end{aligned} \tag{4.10}$$

and we have $a_{j_1k_1}\cdots a_{j_Lk_L}|0,0,1\rangle\!\rangle=y_2^2y_3y_4|0,0,1\rangle\!\rangle$, and we have $\mathrm{Tr}(a_{j_1k_1}\cdots a_{j_Lk_L})=y_2^2y_3y_4$.

5 Concluding remarks

In this work, we applyed the generalized matrix product Ansatz to represent the stationary weights of the inhomogeneous N-species TASEP. We also explained that our solution to the Ansatz is equivalent to Ayyer and Linusson's combinatorial algorithm. Our analysis was motivated by some conjectures proposed by Lam and Williams [12]. For example, for $L=N+1$, they claim that the stationary probability is a polynomial with respect to the hoping rates with non-negative integer coefficients and is a non-negative integral sum of Schubert

polynomials. The first observation is an obvious consequence of Kirchhoff's matrix tree formula [18]. (Note that this is also an outcome of our matrix product solution because all operators have positive entries.) However, the relation with Schubert polynomials is still unclear. Another interesting problem would be to extend the present study to the partially asymmetric case and to classify the multi-species systems with arbitrary inhomogeneous hopping rates that can be solved by the matrix product Ansatz. Finally, we note here that the inhomogeneities are linked to the particles rather than to the underlying lattice. Extending our approach to models with lattice defects (such as the Janowsky and Lebowitz model in which the insertion of a slow bond can generate a shock [9]) remains a very challenging open question.

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A Proof of equation (3.17)

Here we show the first case (3.17) of the algebra.

The case when $\boxed{j \neq k, j' \leq N}$. We have

$$a_{j\kappa} = 0 \quad \text{or} \quad j\text{th component of } a_{j\kappa} \text{ is } \delta \quad (\text{for } \kappa = k, k'), \tag{A.1}$$

$$a_{j'\kappa} = 0 \quad \text{or} \quad j\text{th component of } a_{j'\kappa} \text{ is } A \quad (\text{for } \kappa = k, k'). \tag{A.2}$$

In any cases, we find $a_{jk}a_{j'k'} = a_{j'k}a_{jk'} = 0$, and thus the left-hand side is 0. The right-hand side is also 0 thanks to $\widehat{a}_{jk} = \widehat{a}_{j'k'} = 0$.

The case when $\boxed{j = k, j' \leq N}$. The left hand side is $-x_j a_{jk'} a_{j'j} = 0$ thanks to $a_{j'j} = 0$. The right-hand side is again 0 thanks to $\widehat{a}_{jk} = \widehat{a}_{j'k'} = 0$.

The case when $\boxed{j < k, j' = N+1}$. Since

$$a_{jk}a_{j'k'} = a_{jk'}a_{j'k} = y_j A^{\otimes(j-1)} \otimes \delta \otimes \mathbb{1}^{\otimes(k-j-1)} \otimes \epsilon \otimes \mathbb{1}^{\otimes(k'-k-1)} \otimes \epsilon \otimes \mathbb{1}^{\otimes(N-k'-1)}, \tag{A.3}$$

the left hand side is calculated as

$$-A^{\otimes(j-1)} \otimes \delta \otimes \mathbb{1}^{\otimes(k-j-1)} \otimes \epsilon \otimes \mathbb{1}^{\otimes(k'-k-1)} \otimes \epsilon \otimes \mathbb{1}^{\otimes(N-k'-1)}, \tag{A.4}$$

which agrees with the right-hand side. (We read $\cdots \mathbb{1}^{\otimes(N-k-1)} \otimes \epsilon \otimes \mathbb{1}^{\otimes(-1)} = \cdots \mathbb{1}^{\otimes(N-k-1)}$ for $k' = N$.)

The case when $\boxed{j = k, j' = N+1}$. The left-hand side is calculated as

$$-x_j a_{jk'} a_{j'j} = -A^{\otimes(j-1)} \otimes \mathbb{1}^{\otimes(k'-j)} \epsilon \otimes \mathbb{1}^{\otimes(N-k'-1)}, \tag{A.5}$$

which agrees with the right-hand side.

The case when $\boxed{j > k, j' = N+1}$. We have $a_{jk}a_{j'k'} = 0$ thanks to $a_{jk} = 0$. Since the kth component of each term of $a_{j'k}$ is ϵ, and $a_{jk'}$=0 or the kth component $a_{jk'}$ is A, we also have $a_{jk'}a_{j'k} = 0$. Thus the left hand side is 0. The right hand side is also 0 thanks to $a_{jk} = \widehat{a}_{jk} = 0$.

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